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EXACT CONTROLLABILITY FOR THE WAVE EQUATION WITH VARIABLE COEFFICIENTS

BY

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ABSTRACT

We consider in this paper the evolution system $y'' - Ay = 0$, where $A = \partial_i(a_{ij}\partial_j)$ and $a_{ij} \in C^1(\mathbb{R}^+; W^{1,\infty}(\Omega)) \cap W^{1,\infty}(\Omega \times \mathbb{R}^+)$, with initial data given by $(y_0, y_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ and the nonhomogeneous condition $y = v$ on $\Gamma \times]0, T[$. Exact controllability means that there exist a time $T > 0$ and a control v such that $y(T, v) = y'(T, v) = 0$. The main result of this paper is to prove that the above system is exactly controllable when T is "sufficiently large". Moreover, we obtain sharper estimates on T.

1. Introduction and statement of the results

Let Ω be a non-empty, bounded, open set in $\mathbb{R}^n$ ($n \in \mathbb{N}^*$), with boundary Γ of class C^2, and denote $\nu(x) = (\nu_1(x), \dots, \nu_n(x))$ the outward unit normal vector to Γ.

Let $A = \partial_i(a_{ij}\partial_j)$ be a second-order elliptic differential operator with coefficients $a_{ij} \in C^1(\mathbb{R}^+; W^{1,\infty}(\Omega)) \cap W^{1,\infty}(\Omega \times \mathbb{R}^+)$ where $\mathbb{R}^+ = [0, +\infty)$ (throughout this paper we use the summation convention for repeated indices), such that

$$(1.1) \qquad a_{ij} = a_{ji} \quad \text{and} \quad a_{ij}\xi_i\xi_j \geq \alpha|\xi|^2 \quad \text{in } \bar{\Omega} \times \mathbb{R}^+$$

for some $\alpha > 0$ and for all $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$.

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Consider the problem

$$(1.2) \qquad \begin{cases} y'' - Ay = 0 & \text{in } \Omega \times \mathbb{R}^+, \\ y = v & \text{on } \Gamma \times \mathbb{R}^+, \\ y(0) = y_0 \text{ and } y'(0) = y_1 & \text{in } \Omega, \end{cases}$$

where $' = \partial/\partial t$ and $y(0)$ and $y'(0)$ denote, respectively, the functions $x \longmapsto y(x,0)$ and $x \longmapsto y'(x,0)$.

The problem of exact controllability for system (1.2) states as follows: given $T > 0$ large enough, is it possible, for every initial data (y_0, y_1) from a suitable space, to find a corresponding control v deriving the system to rest at time T, i.e., such that the solution y of (1.2) satisfies

$$y(T) = y'(T) = 0 \quad \text{in } \Omega?$$

Concerning the exact controllability for system (1.2) we note that the case where a_{ij} is independent of the time t was studied by Avellaneda and Lin [1], Ho [3, 4], Komornik [5] and Lions [7], and the case $a_{ij} = \delta_{ij}a(t)$ was studied by Munõz Rivera [9]. The objective of this paper is to show that this system is exactly controllable in the general case. For that we use the Hilbert Uniqueness Method (HUM) introduced by Lions [8] and a new approach given by Guesmia [2].

Let us define u as the solution of the following system:

$$(1.3) \qquad \begin{cases} u'' - Au = 0 & \text{in } \Omega \times \mathbb{R}^+, \\ u = 0 & \text{on } \Gamma \times \mathbb{R}^+, \\ u(0) = u_0 \text{ and } u'(0) = u_1 & \text{in } \Omega. \end{cases}$$

The well-posedness of the problem (1.3) can be established by standard methods of evolution systems (cf. Pazy [10], ch. 5); we omit the details. This problem is well-posed in the following sense:

* For every $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, the system (1.3) has a unique solution (defined in a suitable weak sense) satisfying

$$u \in C(\mathbb{R}^+; H_0^1(\Omega)) \cap C^1(\mathbb{R}^+; L^2(\Omega)),$$

where $H_0^1(\Omega) = \{v \in H^1(\Omega) \colon v = 0 \quad \text{on } \Gamma\}$.

* If $(u_0, u_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$ then the solution (called a strong solution) is more regular:

$$u \in C(\mathbb{R}^+; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1(\mathbb{R}^+; H_0^1(\Omega)) \cap C^2(\mathbb{R}^+; L^2(\Omega)).$$

* The energy of the (weak) solution, defined by the formula

$$E(t) = \frac{1}{2}\int_\Omega (\left|u'\right|^2 + a_{ij}\partial_i u \partial_j u)dx, \quad t \in \mathbb{R}^+, \tag{1.4}$$

is a positive function and satisfies the identity

$$E'(t) = \frac{1}{2}\int_\Omega a'_{ij}\partial_i u \partial_j u dx, \quad \forall t \in \mathbb{R}^+; \tag{1.5}$$

then our system is not conservative (that is, the energy is not a constant function) in general. The observability inequalities (1.13) below are generally more challenging to achieve than in the case of conservative systems as in [3, 4, 5, 7].

Fix a point $x^0 \in \mathbb{R}^n$ and, putting $m(x) = x - x^0$, $R = \|m\|_{L^\infty(\Omega)}$. We determine the sets

$$\Gamma_0 = \{x \in \Gamma : m(x).\nu(x) \le 0\} \quad \text{and} \quad \Gamma_1 = \Gamma \smallsetminus \Gamma_0. \tag{1.6}$$

(For example, we may always choose $\Gamma_0 = \emptyset$ and $\Gamma_1 = \Gamma$.)

Remark: As an example of the existence of such a point x^0 we can consider the following domain: $\Omega = \Omega_1 \smallsetminus \bar{\Omega}_0$ where Ω_1 and Ω_0 are two open domains star chapped with respect to x^0, with boundary Γ_1 and Γ_0, respectively, and $\bar{\Omega}_0 \subset \Omega_1$. By our choice of x^0 we easily get that the hypotheses on Γ_1 and Γ_0 hold.

Let $\gamma, \beta > 0$ and $\lambda \ge 0$ denote the real numbers such that

$$(2a_{ij} - m_k \partial_k a_{ij})\xi_i \xi_j \ge \gamma a_{ij}\xi_i\xi_j \quad \text{in } \Omega \times \mathbb{R}^+, \tag{1.7}$$

$$(m_k \xi_k)^2 \le \beta a_{ij}\xi_i\xi_j \quad \text{in } \Omega \times \mathbb{R}^+ \tag{1.8}$$

and

$$\left|a'_{ij}\xi_i\xi_j\right| \le \lambda a_{ij}\xi_i\xi_j \quad \text{in } \Omega \times \mathbb{R}^+ \tag{1.9}$$

for all $\xi \in \mathbb{R}^n$. (Because Ω is a bounded set, $a_{ij} \in W^{1,\infty}(\Omega \times \mathbb{R}^+)$ and, by (1.1), then β and λ exist always.) Assume that

$$\gamma \le 2n, \tag{1.10}$$

$$\frac{4\lambda\sqrt{\beta}}{\gamma} < 1 \tag{1.11}$$

and let $T \in \mathbb{R}^+$ such that

$$T > -\frac{1}{\lambda}\log\left(1 - \frac{4\lambda\sqrt{\beta}}{\gamma}\right). \tag{1.12}$$

If $\lambda = 0$, then we take $T > 4\sqrt{\beta}/\gamma$.

First, we will prove the following observability result.

THEOREM 1.1: *Assume* (1.1), (1.6), (1.10)–(1.12). *Then there exist two positive constants* c_1 *and* c_2 *such that every strong solution of* (1.3) *satisfies the inequalities*

$$c_1 E(0) \le \int_0^T \int_{\Gamma_1} a_{ij}\partial_i u \partial_j u d\Gamma dt \le c_2 E(0). \tag{1.13}$$

Remarks: * Applying the same density argument as before, the estimates (1.13) remain valid for weak solutions, too, because for every weak solution of (1.3), the second estimate in (1.13) allows us to define the trace of $a_{ij}\partial_i u \partial_j u$ on $\Gamma_1 \times \mathbb{R}^+$ as an element of $L^2_{\text{loc}}(\Gamma_1 \times \mathbb{R}^+)$.

* If $\lambda = 0$ (i.e. $a'_{ij} = 0$) then the condition on T reduces to $T > 4\sqrt{\beta}/\gamma$. In this case, we obtain a better condition on T than the one given by Ho [4].

* The first inequality in (1.13) cannot hold for arbitrarily small T. In [5], Komornik shows that the condition $T > 4\sqrt{\beta}\gamma$ is the best possible if

$$\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$$

and $A = \Delta$ (i.e., $a_{ij} = \delta_{ij}$, $\lambda = 0$, $\gamma = 2$).

* The observability inequalities (1.13), which yield energy decay (stabilization) results, will be proved under the restrictive condition (1.7). This condition has been assumed for the first time in Komornik [5]. The general case remains open. On the other hand, it is possible to consider more general conditions than (1.7) and (1.9); we take $\gamma(t)$ and $\lambda(t)$ as two functions on time t. To keep this paper from becoming too long, we consider only the case of (1.7) and (1.9).

* Theorem 1.1 means that in some sense the observation of the solution in a neighbourhood of the boundary during a sufficiently large time allows one to determine the initial data. Indeed, if two solutions coincide in this set, then the boundary integral in (1.13) for their difference vanishes and therefore the initial energy of their difference is equal to zero by the first inequality in (1.13). From the unicity of the solution, this implies that the two solutions correspond in fact to the same initial data and hence they are identical.

Applying the Hilbert Uniqueness Method (HUM) introduced by Lions [6, 8] we shall deduce from Theorem 1.1 an exact controllability result for the system (1.2).

THEOREM 1.2: *Assume* (1.1), (1.6), (1.10)–(1.12). *Then for any given* $(y_0, y_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ *there exists a corresponding control function* $v \in L^2_{loc}(\mathbb{R}^+; L^2(\Gamma))$ *such that the solution of* (1.2) *satisfies*

$$y(T) = y'(T) = 0 \quad in\ \Omega. \tag{1.14}$$

Moreover, we may assume that v vanishes outside of $\Gamma_1 \times (0,T)$.

Remark: Because our system (1.2) is linear, then Theorem 1.2 implies that, for any given $(y_0, y_1), (\bar{y}_0, \bar{y}_1) \in L^2(\Omega) \times H^{-1}(\Omega)$, there exists a corresponding control function $v \in L^2_{loc}(\mathbb{R}^+; L^2(\Gamma))$ such that the solution of (1.2) satisfies the final condition (see Guesmia [2])

$$y(T) = \bar{y}_0, \quad y'(T) = \bar{y}_1 \quad \text{in } \Omega.$$

2. Observability: Proof of Theorem 1.1

First we prove the following lemma.

LEMMA 2.1: *We have*

$$-\lambda E(t) \leq E'(t) \leq \lambda E(t) \quad \forall t \in \mathbb{R}^+, \tag{2.1}$$

$$e^{-\lambda t} E(0) \leq E(t) \leq e^{\lambda t} E(0) \quad \forall t \in \mathbb{R}^+. \tag{2.2}$$

Proof: From (1.4), (1.5) and (1.9) we obtain (2.1). By Gronwall's inequality we deduce (2.2) from (2.1). Hence the lemma follows.

Now we prove Theorem 1.1. Fix an arbitrary function $h \in (W^{1,\infty}(\Omega))^n$ and a number $T > 0$. We deduce from (1.3) that

$$\begin{aligned} 0 = &\int_0^T \int_\Omega (h_k \partial_k u)(u'' - \partial_i(a_{ij}\partial_j u))dxdt \\ = &\Big[\int_\Omega h_k \partial_k u u' dx\Big]_0^T - \int_0^T \int_\Gamma h_k \partial_k u a_{ij} \partial_j u \nu_i d\Gamma dt \\ &+ \int_0^T \int_\Omega (\partial_i h_k a_{ij} \partial_j u \partial_k u + h_k a_{ij} \partial_j u \partial_i \partial_k u - \frac{1}{2} h_k \partial_k |u'|^2) dxdt. \end{aligned}$$

Since (using the symmetry of a_{ij})

$$a_{ij}\partial_j u \partial_i \partial_k u = \frac{1}{2}\partial_k(a_{ij}\partial_j u \partial_i u) - \frac{1}{2}(\partial_k a_{ij})\partial_i u \partial_j u, \tag{2.3}$$

integrating by parts the last two terms in the last integral and then multiplying by 2, we obtain the following identity:

$$\begin{aligned} &\int_0^T \int_\Gamma (2h_k \partial_k u a_{ij} \partial_j u \nu_i + (h.\nu)(|u'|^2 - a_{ij}\partial_i u \partial_j u)) d\Gamma dt \\ = &\Big[\int_\Omega 2h_k \partial_k u u' dx\Big]_0^T - \int_0^T \int_\Omega h_k \partial_k a_{ij} \partial_i u \partial_j u dxdt \\ &+ \int_0^T \int_\Omega (2\partial_i h_k a_{ij} \partial_j u \partial_k u + (\operatorname{div} h)(|u'|^2 - a_{ij}\partial_i u \partial_j u)) dxdt. \end{aligned} \tag{2.4}$$

Using the assumption $h_k \in W^{1,\infty}(\Omega)$, $a_{ij} \in W^{1,\infty}(\Omega \times \mathbb{R}^+)$, the condition (1.1) and the estimate (2.2), the right-hand side of (2.4) can be easily majorized by $cE(0)$, where c is a positive constant. Furthermore, we deduce from the homogeneous Dirichlet boundary condition in (1.3) that

$$u' = 0 \quad \text{and} \quad \partial_k u \nu_i = \partial_\nu u \nu_k \nu_i = \partial_i u \nu_k \quad \text{on } \Gamma,$$

and hence

$$h_k \partial_k u a_{ij} \partial_j u \nu_i = (h.\nu) a_{ij} \partial_i u \partial_j u.$$

Therefore the left-hand side of (2.4) reduces to

$$\int_0^T \int_\Gamma (h.\nu) a_{ij} \partial_i u \partial_j u d\Gamma dt.$$

Choosing h such that $h = \nu$ on Γ, the second inequality in (1.13) follows with $c_2 = c$.

Choosing now $h(x) = m(x)$ the identity (2.4) reduces to

$$\begin{aligned}\int_0^T \int_\Gamma &(h.\nu) a_{ij} \partial_i u \partial_j u d\Gamma dt \\ &= \Big[\int_\Omega 2h_k \partial_k u u' dx\Big]_0^T + \int_0^T \int_\Omega ((2-n) a_{ij} \partial_i u \partial_j u + n|u'|^2) dx dt \\ &\qquad - \int_0^T \int_\Omega h_k \partial_k a_{ij} \partial_i u \partial_j u dx dt.\end{aligned}$$

Furthermore, we also deduce from (1.3) that

$$\begin{aligned}0 &= \int_0^T \int_\Omega u(u'' - \partial_i(a_{ij}\partial_j u)) dx dt \\ &= \Big[\int_\Omega u u' dx\Big]_0^T - \int_0^T \int_\Gamma u a_{ij} \partial_j u \nu_i d\Gamma dt + \int_0^T \int_\Omega (a_{ij} \partial_i u \partial_j u - |u'|^2) dx dt \\ &= \Big[\int_\Omega u u' dx\Big]_0^T + \int_0^T \int_\Omega (a_{ij} \partial_i u \partial_j u - |u'|^2) dx dt.\end{aligned}$$

Multiplying this equality by $n - \gamma/2$ and combining with the preceding identity we obtain that

$$\begin{aligned}\int_0^T \int_\Gamma &(h.\nu) a_{ij} \partial_i u \partial_j u d d t \\ &= \Big[\int_\Omega (2h_k \partial_k u + (n - \frac{\gamma}{2}) u) u' dx\Big]_0^T - \int_0^T \int_\Omega h_k (\partial_k a_{ij}) \partial_i u \partial_j u dx dt \\ &\quad + \int_0^T \int_\Omega ((2 - \frac{\gamma}{2}) a_{ij} \partial_i u \partial_j u + \frac{\gamma}{2} |u'|^2) dx dt\end{aligned}$$

since, using (1.7), (1.6) and (1.4), we obtain

(2.5)
$$R\int_0^T\int_{\Gamma_1} a_{ij}\partial_i u\partial_j u d\Gamma dt \geq \gamma\int_0^T E(t)dt - \Big|\Big[\int_\Omega (2h_k\partial_k u + (n-\frac{\gamma}{2})u)u'dxdt\Big]_0^T\Big|.$$

Let us majorize the last integral. We have

$$\begin{aligned}
&\|2h_k\partial_k u + (n-\gamma/2)u\|^2_{L^2(\Omega)} - \|2h_k\partial_k u\|^2_{L^2(\Omega)}\\
&= \int_\Omega((n-\gamma/2)^2u^2 + 4(n-\gamma/2)h_k\partial_k uu)dx\\
&= \int_\Omega((n-\gamma/2)^2u^2 - 2(n-\gamma/2)nu^2)dx + \int_\Gamma 2(n-\gamma/2)h_k\nu_k u^2 d\Gamma\\
&= (\gamma^2/4 - n^2)\int_\Omega u^2dx \leq 0
\end{aligned}$$

(cf. (1.10)). Therefore, using (1.8) and (1.4) we obtain

$$\begin{aligned}
\Big|\int_\Omega(2h_k\partial_k u + (n-\frac{\gamma}{2})u)u'dx\Big| &\leq \frac{1}{4\sqrt{\beta}}\int_\Omega(2h_k\partial_k u)^2dx + \sqrt{\beta}\int_\Omega|u'|^2dx.\\
&\leq \sqrt{\beta}\int_\Omega(a_{ij}\partial_i u\partial_j u + |u'|^2)dx = 2\sqrt{\beta}E(t). \qquad (2.6)
\end{aligned}$$

Therefore we deduce from (2.5) and (2.6) the inequality

$$R\int_0^T\int_{\Gamma_1} a_{ij}\partial_i u\partial_j u d\Gamma dt \geq \gamma\int_0^T E(t)dt - 2\sqrt{\beta}(E(0)+E(T)). \qquad (2.7)$$

Suppose that $E(T) \geq E(0)$. Now from (2.1) we have

$$E(t) \geq \frac{1}{\lambda}((1-e^{-\lambda t})E(t))', \quad \forall t \geq 0.$$

Then from (2.7) we obtain

$$\begin{aligned}
R\int_0^T\int_{\Gamma_1} a_{ij}\partial_i u\partial_j u d\Gamma dt &\geq \Big(\frac{\gamma}{\lambda}(1-e^{-\lambda T}) - 4\sqrt{\beta}\Big)E(T)\\
&\geq \Big(\frac{\gamma}{\lambda}(1-e^{-\lambda T}) - 4\sqrt{\beta}\Big)E(0)
\end{aligned}$$

and the first estimate of (1.13) follows with

$$c_1 = \frac{\gamma}{R\lambda}(1-e^{-\lambda T}) - \frac{4}{R}\sqrt{\beta}.$$

(From (1.12) we have $c_1 > 0$.) With the same reasoning we can argue the case for $E(T) \leq E(0)$, and the proof is thus complete.

We give now an equivalent form of the integral in (1.13).

LEMMA 2.2: *Assume* (1.1) *and put*

$$\eta = \sum_{i,j} \|a_{ij}\|^2_{L^\infty(\Gamma\times\mathbb{R}^+)}.$$

Then every strong solution of (1.3) *satisfies on* $\Gamma \times \mathbb{R}^+$ *the inequalities*

$$\alpha a_{ij}\partial_j u \partial_i u \le \left|a_{ij}\partial_j u \nu_i\right|^2 \le \frac{\eta}{\alpha} a_{ij}\partial_j u \partial_i u. \tag{2.8}$$

Applying the same density argument as before, the estimates (2.8) remain valid for the weak solution, too.

Proof: The proof of the second inequality does not use the boundary condition:

$$\left|a_{ij}\partial_j u\nu_i\right|^2 \le \sum_i \left|a_{ij}\partial_j u\right|^2 \le \Big(\sum_{i,j}\left|a_{ij}\right|^2\Big)\Big(\sum_j \left|\partial_j u\right|^2\Big) \le \frac{\eta}{\alpha} a_{ij}\partial_j u\partial_i u.$$

For the proof of the reverse inequality, first we note that thanks to the boundary conditions in (1.3) we have

$$\begin{aligned} a_{ij}\partial_j u\partial_i u &= (a_{ij}\partial_j u\nu_i)\partial_\nu u \le \left|a_{ij}\partial_j u\nu_i\right|\left|\partial_\nu u\right| \\ &\le \left|a_{ij}\partial_j u\nu_i\right|(\left|\nabla u\right|^2)^{\frac{1}{2}} \le \left|a_{ij}\partial_j u\nu_i\right|\Big(\frac{1}{\alpha}a_{ij}\partial_j u\partial_i u\Big)^{\frac{1}{2}} \end{aligned}$$

and the first inequality in (2.8) follows.

3. Controllability: Proof of Theorem 1.2

The main idea is to seek a control in the form

$$v = \begin{cases} a_{ij}\partial_j u\nu_i & \text{on } \Gamma_1\times]0,T[\\ 0 & \text{on } \Gamma\times\mathbb{R}^+ \setminus \Gamma_1\times]0,T[\end{cases} \tag{3.1}$$

where u is the solution of (1.3) for some suitable initial data. Thanks to Theorem 1.1 and Lemma 2.2 these controls have the required regularity in Theorem 1.2 for all weak solutions of (1.3). Let us define z as the solution of

$$\begin{cases} z'' - Az = 0 & \text{in } \Omega\times\mathbb{R}^+, \\ z = v & \text{on } \Gamma\times\mathbb{R}^+, \\ z(T) = z'(T) = 0 & \text{in } \Omega. \end{cases} \tag{3.2}$$

It is well known that $v \in L^2(\Gamma\times\mathbb{R}^+)$ for all weak solutions u of (1.3) and system (3.2) has only one solution, $z \in C(\mathbb{R}^+; L^2(\Omega))$. Let us multiply the equation in

(1.3) by z and integrate by parts formally. We obtain

$$\begin{aligned} 0 &= \int_0^T \int_\Omega z(u'' - \partial_i(a_{ij}\partial_j u))dxdt \\ &= \int_0^T \int_\Omega (z'' - \partial_i(a_{ij}\partial_j z))udxdt + \left[\int_\Omega (zu' - z'u)dx\right]_0^T \\ &\quad + \int_0^T \int_\Gamma (-za_{ij}\partial_j u\nu_i + ua_{ij}\partial_j z\nu_i)d\Gamma dt; \end{aligned}$$

using definition (3.1) of v and systems (1.3) and (3.2), we obtain

$$\int_0^T \int_{\Gamma_1} \left|a_{ij}\partial_j u\nu_i\right|^2 d\Gamma dt = \int_\Omega (z'(0)u_0 - z(0)u_1)dx.$$

Hence, putting

$$H = H_0^1(\Omega) \times L^2(\Omega) \quad \text{and} \quad H' = H^{-1}(\Omega) \times L^2(\Omega)$$

and setting

$$\Lambda(u_0, u_1) = (z'(0), -z(0))$$

for brevity, we have

$$\langle \Lambda(u_0, u_1), (u_0, u_1)\rangle_{H',H} = \int_0^T \int_{\Gamma_1} \left|a_{ij}\partial_j u\nu_i\right|^2 d\Gamma dt. \tag{3.3}$$

Obviously, Λ: $H \to H'$ is a bounded linear map. Applying HUM, it is sufficient to show that Λ is onto. Indeed, then for any given $(y_0, y_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ it will suffice to choose the control v defined by (3.1), where u is the solution of (1.3) corresponding to $(u_0, u_1) = \Lambda^{-1}(y_1, -y_0)$, and then the function z defined by system (3.2) satisfies $(z(0), z'(0)) = (y_0, y_1)$, taking v defined by (3.1) in system (1.2). By uniqueness of solutions for linear hyperbolic systems, we conclude that $y = z$, and from (3.2) the result (1.14) follows.

Applying the first estimate of (1.13) in Theorem 1.1 and Lemma 2.2, we conclude from identity (3.3) that Λ is coercive. Applying the Lax–Milgram theorem we conclude that Λ is an isomorphism.

The proof of Theorem 1.2 is now complete.

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